Stability of a Coastal Upwelling Front

2. Model Results and Comparison With Observations

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A two-layer shallow water equation model is used to investigate the linear stability of a coastal upwelling front. The model features a surface front parallel to a coastal boundary and bottom topography which is an arbitrary function of the cross-shelf coordinate. The existence of unstable waves on the model coastal upwelling front, as suggested by the general stability theorem developed in a companion paper, is confirmed by solving directly the linearized equations of motion. The unstable wave motions are frontally trapped and dominant in the upper layer. The wave propagates phase in the direction of the basic state flow, and the primary energy conversion is via baroclinic instability. The effect of varying the model parameters is presented. Moving the front closer than \( \sim 2 \) Rossby radii to the coastal boundary results in a decrease in the growth rate of the fastest growing wave. Increasing the overall vertical shear of the basic state flow, by either decreasing the lower layer depth or increasing the steepness of the interface, results in an increase in the growth of the fastest growing wave. A bottom sloping in the same sense as the interface results in a decrease of the growth rates and alongfront wave numbers of the unstable waves in the system. Linearized bottom friction is included in the stability model and results in a decrease in the growth rates of the unstable waves by extracting energy from the system. Since the unstable mode is strongest in the upper layer, bottom friction will not stabilize the upwelling front. A comparison between the predictions from the simple two-layer model and observed alongfront variability for three areas of active upwelling is presented. Reasonable agreement is found, suggesting that the observed alongfront variability can be interpreted in terms of the instability of a coastal upwelling front.

1. INTRODUCTION

Upwelling fronts are important features in coastal ocean circulation. They form when an upwelling wind directed so that the coastal barrier is to its left in the northern hemisphere drives an offshore Ekman flux in the upper part of the water column. The sharp, horizontal near-surface density contrast between cold, saline water, which has upwelled offshore to replace the water fluxed offshore in the Ekman layer, and the less dense surface water offshore is called the coastal upwelling front. Upwelling occurs in many of the world's coastal oceans, and the upwelling front plays an important role (e.g., separating nutrient-rich water inshore from nutrient-poor water offshore) in the biology of the coastal environment. Observations often show a great deal of alongshore variability in the offshore position of the front. These meanders in the front directly the normally alongshore coastal flow cross-shelf. Observations suggest that the meanders may grow large enough to break the continuity of the alongshore flow by forming detached eddies. Other observations show meander peaks away from the coast growing in the offshore direction, creating strong cross-shore flows well seaward of the continental shelf break (\( \sim 30 \) km offshore on the western coast of the United States). This would serve to transport large amounts of cold, nutrient-rich upwelled water off the shelf. These so-called offshore jets or squirts have been documented in a number of studies off the western coast of the United States [e.g., Davis, 1985a; Kosro, 1987].

In the first part of this study [Barth, this issue] (hereinafter referred to as part 1), a model of the coastal upwelling front is developed and analyzed in order to answer the question: Can wave-like disturbances with the properties of observed frontal variability be produced by hydrodynamic instability? The governing equations of motion employed are the shallow water equations rather than the quasi-geostrophic equations [Pedlosky, 1986] because the latter, while simplifying the instability calculations, are inapplicable to frontal regions. Large interface displacements, strong horizontal shears, and large slopes in the bottom topography (which are allowed in this study) are not allowed in quasi-geostrophic theory.

The model geometry is shown in Figure 1a. The model has a rigid lid and is on an f plane. Two homogeneous layers of density \( \rho_1 \) and \( \rho_2 \) (\( \rho_2 > \rho_1 \)) lie adjacent to a coastal barrier. The origin of the coordinate system is chosen to be the coast with \( x, y, \) and \( z \) being the cross-front (positive onshore), alongfront, and vertical directions. The entire system is rotating about the \( z \) axis with an angular frequency \( \Omega \), where \( f \) is the Coriolis parameter. The layer thicknesses are denoted by \( h_1 \) and \( h_2 \), while the bottom topography, which is an arbitrary function of \( x \) but assumed uniform in \( y \), is given by \( H = h_1 + h_2 \). The sea surface elevation is denoted by \( \zeta \). The surface front, modeled as the interface between the layers of different densities, lies parallel to the coast at the point \( (x, y) = (x_f, 0) \), offshore of the coastal barrier. The sloping interface and bottom adjacent a distant flat-bottom region (representing the deep ocean offshore of the continental margin) with constant layer depths \( (H_1, H_2) \). A basic alongfront flow \( (u) \) which is uniform in \( y \), in-
dependent of time \( t \), and in geostrophic balance exists in the upper layer (Figure 1b). For simplicity there is no basic state flow in the lower layer.

In part 1, the governing equations are solved in their nondimensional form. The scales needed to redimensionalize the results presented here are

\[
\begin{align*}
(x, y) &= R(x, y) \\
(u_1, v_1, u_2, v_2) &= (g'H_1)^{1/2}(u_1, v_1, u_2, v_2) \\
\zeta_1 &= \delta H_1 \zeta_1
\end{align*}
\]

The variables subscripted with an asterisk are dimensional, and the numerical subscripts on the velocity components \((u, v)\) indicate either the upper or lower layer. Horizontal length is scaled by the internal Rossby radius of deformation, \( R = (g'H_1)^{1/2}/f \), where the reduced acceleration due to gravity is given by \( g' = g(\rho_2 - \rho_1)/\rho_2 = g\delta \). Note that the density defect, \( \delta \), is much less than 1. For typical coastal upwelling fronts \( R \sim 5 - 10 \) km. Velocities are nondimensionalized by the internal gravity wave phase speed, \((g'H_1)^{1/2}\), which is typically \( 50 - 100 \) cm s\(^{-1}\). Time is scaled by \( f^{-1} \).

In part 1 global conservation statements for energy, momentum, and potential vorticity are combined to yield a theorem which allows the a priori determination of the stability of a coastal upwelling front. The stability criteria differ from those obtained from quasi-geostrophic theory. The theorem successfully encompasses the results of many previous frontal models [Paldor, 1983; Killworth, 1983]. When applied to the flows of interest here, the theorem suggests that coastal upwelling fronts are potentially unstable no matter what the basic state configuration is. Potential instability is implied by satisfaction of necessary conditions for instability. It is still essential to verify that unstable waves do exist on a coastal upwelling front, and this is done by solving numerically (see the appendix for details) the governing equations of motion.

The remainder of this paper is organized as follows. First, results are presented for a model with an inviscid basic state flow and a flat bottom. In particular, results for a model front located far from a coastal barrier and with uniform upper layer basic state potential vorticity are compared to the results of Killworth et al. [1984]. The effects on the properties of the unstable waves produced by varying the total depth, the distance to the coastal wall, and the shape of the interface profile are discussed. For one standard case, the energy transfers in the system are analyzed in detail and the conservation statements of part 1, section 3, confirmed. The effect of a strongly sloping bottom is presented in section 2.2. Next, linearized bottom friction is included and its influence on the unstable waves discussed. In section 3 a comparison of the model calculations to observations from three upwelling regions is presented. Finally, the results are discussed and conclusions summarized.

2. RESULTS

2.1 INVIScid, FLAT-BOTTOM MODEL

In this section, the stability of a front in a simple model geometry, a flat bottom adjacent to a coastal wall, is investigated. Results from a model with sloping bottom topography are presented in the next section. The choice of the basic state interface profile (and hence the basic state flow via geostrophy) is intended to represent a fully mature coastal upwelling front. Subsurface density observations may be used to specify an interface profile. However, this information, in particular, vertical sections perpendicular to the front, is often not available for the coastal upwelling front of interest (e.g., one that is apparent in a satellite infrared image). When observations are available, the presence of small-scale density features due to internal waves, mixing, and mixed-layer dynamics complicates the choice of a basic state interface profile. Further, observations encompassing the frontal region are never truly synoptic. In light of the above discussion, an instability model whose inputs (e.g., layer depths, density difference between layers) can be set using a few hydrographic casts or historical data is desirable.

The approach used here is to consider interface profiles which arise from simple models of the coastal upwelling process. In these models [e.g., Csanady, 1971; Pedlosky, 1978], the scale over which the interface or density surfaces warp upward to contact the sea surface (in an approximately exponential shape) is generally the internal Rossby radius of deformation (typically 5-10 km). A model which allows entrainment between layers due to wind mixing [de Szaeke and Richman, 1984] describes the offshore migration of the surface front, a feature which is observed in nature. Therefore the choice for the interface profile is a family of exponentials given by

\[
\begin{align*}
\hat{h}_1(x) &= 1 - \exp[\alpha(x - x_f)] & x \leq x_f \\
\hat{h}_1(x) &= 0 & x > x_f
\end{align*}
\]

from which the upper layer basic state alongfront flow is

\[
\bar{\nu}(x) = \frac{\partial}{\partial z} [\hat{h}_1(x)] = -\alpha \exp[\alpha(x - x_f)] & x \leq x_f
\]

A uniform potential vorticity basic state has \( \alpha = 1.0 \). For \( \alpha < 1.0 \) the interface is less steeply sloping than the uniform
potential vorticity front (e-folding length greater than 1.0, which in dimensional units is the internal Rossby radius of deformation), and for $\sigma > 1.0$ the front is more steeply sloping (e-folding length less than $R$). Interface profiles evident in vertical sections of density for many coastal upwelling fronts appear to be realistically represented by the exponential form given above.

The stability of a flat-bottom model with uniform basic state upper layer potential vorticity is investigated first. Since no mean flow exists in the lower layer, the potential vorticity there is dictated by the change in layer depth. The presence of unstable waves in a system with this distribution of potential vorticity (i.e., lack of a change in sign of the potential vorticity gradient) will distinguish these ageostrophic flows from flows governed by traditional quasi-geostrophic dynamics (see part 1, section 3). The depth of the upper layer far from the surface front ($H_1$ in Figure 1) is set equal to 1 for all the models presented here. Using this, the total depth $H = H_1 + H_2$ can be denoted by $r$ so that the lower layer depth is equal to $r - 1$. The stability theorem from part 1 suggests that this basic state will be unstable if the lower layer is sufficiently shallow. The first model geometry considered has equal upper and lower layer depths, $r = 2$, and the surface front is located two Rossby radii from the coastal barrier ($x_f = -2R$). The results from this model geometry and basic state flow configuration will serve as a standard case with which to compare the results for models with different parameter choices.

The complex frequency ($\sigma$), which emerges as the eigenvalue for the problem after seeking normal mode solutions to the governing equations (see the appendix), yields two important properties of the unstable wave. The imaginary part of $\sigma$ is the rate at which the unstable wave grows with time. The growth rate ($\sigma_r$) is plotted as a function of alongfront wave number in Figure 2. No unstable modes exist above $l = l_c \approx 1.50$ where $\sigma_r = 0$. This "short wave cutoff" is related to restricting the vertical shear in the system to an infinitely thin layer at the interface and is a common feature of stability models which use homogeneous fluid layers [Pedlosky, 1986]. The wave with the largest growth rate occurs at $l = l_m = 1.05$, where $\sigma_{m} = 0.09$ (the subscript $m$ identifies the maximum growth rate). Within the limits of the linear, small-amplitude theory presented here, this is the wave most likely to be observed as its amplitude grows most rapidly from an initial condition containing many wave components. As the disturbance grows to finite amplitude, nonlinear effects may alter the properties of the fastest growing wave.

The real part of the eigenvalue ($\sigma_\lambda$) divided by the wave number ($l$) yields the phase speed of the unstable wave ($c_\lambda$) which is plotted as a function of alongfront wave number in Figure 2b. The solid curve for $l < l_c$ is for the unstable mode, whose growth rate is given in Figure 2a. The values for $c_\lambda$ are all negative which correspond to propagation to $-y$. Phase propagation in the downstream direction is consistent with the notion that the phase speed of an unstable wave must lie within the range of the mean flow speeds. This results from so-called "semicircle" theorems which have been rigorously derived for layer models of instability both using quasigeostrophic dynamics [Pedlosky, 1986] and ageostrophic dynamics [Hayashi and Young, 1987]. Since $\overline{v}$ is in the $-y$ direction the unstable wave should propagate in the same direction with a phase speed less than the maximum flow speed. This result is confirmed in Figure 2b, remembering that maximum flow speed occurs at the front ($x = x_f$) and is equal to $-\sigma$, which for the uniform potential vorticity case considered here is equal to $-1$. At $l = l_m$ the real part of the phase speed splits into two stable waves (dashed curves) which for $l < l_c$ had been resonating to produce the unstable mode. In addition to these stable waves, there exist many more stable vorticity modes with $c_\lambda < 0$ which are not plotted in Figure 2b.

It is useful to consider the dimensional properties of the fastest growing wave. The length, time, and velocity scales used for the redimensionalization are defined in section 1. For $R = 5$ km and mid-latitudes ($f \sim 10^{-4}$) the fastest growing wave has a wavelength of $\sim 30$ km ($2\pi R/l_m$), an e-folding period (the time it takes the perturbation to grow to $exp(1)$ of its initial amplitude) of $\sim 1.5$ days ($1/\sigma_{m}$), and a phase speed toward $-y$ of $\sim 3$ cm s$^{-1}$ ($c_\lambda R f$). This standard model, which has not been tuned to represent a coastal upwelling front for any particular geographic location, yields spatial scales for the fastest growing wave which are of the same order as the observed scales of alongfront variability off, for instance, the coast of Oregon (see section 3.1, Figure 17b). Values for growth rates and alongfront phase speeds are more difficult to obtain from observations, but when estimates of these properties are available [e.g., Petrie et al., 1987], they are comparable to the values pre-

![Fig. 2. (a) Growth rate versus alongfront wave number for the unstable mode present on a model front with uniform basic state upper layer potential vorticity. (b) Real part of the phase speed versus alongfront wave number for the unstable mode ($l < 1.5$, solid curve) and for two stable modes ($l \geq 1.5$, dashed curves).](image-url)
dicted by this simple instability model. To make a more detailed comparison between the model-derived properties of the fastest growing wave and observations, it is necessary to understand the effect of varying the basic state flow and model geometry (e.g., including sloping bottom topography). Following a discussion of these effects, a comparison between the model predictions and observed alongfront variability is made in section 3.

The structure of the most unstable wave is displayed in Fig. 3. Properties of the upper layer are presented in Fig. 3a where the dashed line at \( x = -2 \) represents the surface front. The solid and dashed curves represent contours of \( \zeta_1 \), the sea surface elevation, whose maximum value occurs at the front and is normalized to 1.0. For presentation purposes, the perturbation upper layer velocities are scaled such that the vector beneath the legend represents one velocity unit. Lower layer velocities and contours of lower layer “pressure” \( (\zeta_T = (1 - \theta)\zeta_1 + \zeta_2) \) denote perturbation interface displacement) appear in Fig. 3b. The normalization and scaling procedure is identical for the two layers so that comparisons of properties between them can be made.

All the fields are frontally trapped with a cross-front folding scale of \( \sim 1 \pi \). Upper layer velocities exhibit a pattern consistent with a wave-like deflection of the surface front. If the velocity field were completely geostrophic, the contours of \( \zeta_1 \) would serve as streamlines for the flow. The crossing of \( \zeta_1 \) contours by the velocity vectors, especially near the surface front, is indicative of a significant ageostrophic component to the velocity field. Velocities in the lower layer are much weaker and more geostrophic than those in the upper layer. The lower layer velocity and \( \zeta_T \) fields form closed cells with maxima occurring \( \sim 0.75 \pi \) on the two-layer side of the surface front.

As discussed in part 1, section 3, the interpretation of energy transfers within an unstable system can be thought of in two ways. In traditional instability theory the flow of energy to the unstable disturbance from the basic state and vice versa can be analyzed. For non-divergent flow fields, instability requires energy conversion from the basic state to the disturbance (i.e., positive wave energy \( E_i^p \) and negative mean energy \( E_i^m \) in the language of part 1, section 3). For the flows considered here it was shown (see Part 1, section 3) that unstable waves can exist even if the net flow of energy between the basic state and the disturbance is zero or even when the transfer is from the disturbance to the basic state flow creating negative wave energy and positive mean energy. Wave energy can be positive, zero, or negative in this system because its definition

\[
E_i^p = \frac{1}{2} \int \left( \frac{v_1^2 + v_2^2}{v_1} - 2w_1 \zeta_2 + \overline{h_1^2 (u_2^2 + v_2^2)} \right) dA
\]

contains a term \( (-\overline{v_1 \zeta_2}) \) which is not positive definite. The integral is defined over the undisturbed (prior to the displacement of the surface front by the unstable wave) area of the upper layer. Quantities in the integrand with an overbar are basic state quantities: \( h_1 \) and \( h_2 \) are the upper and lower layer depths, respectively, and \( \overline{v} \) is the alongfront upper layer flow. The other variables are the perturbation velocities in the two layers \( (u_1, v_1, u_2, v_2) \) and the perturbation interface displacement \( (\zeta_2) \). For this term to be negative the correlation between alongfront perturbation velocity and perturbation interface displacement must be such that the disturbance increases the total upper layer thickness where it decreases the total alongfront speed and vice versa (see Part 1, section 3 and Fig. 3, for a more thorough discussion). This term is not present in the definition of wave energy in quasi-geostrophic theory because deviations of the interface from its basic state value are assumed small. These cases are certainly countervailuative, but as described in Part 1, section 3, it is not difficult to find physical systems which can be unstable without the flow of energy from the basic state to the disturbance (e.g., an inverted pendulum).

An expression can be derived (see Part 1, section 3) which relates the time rate of change of the wave energy to the dispersion of particles within a basic state potential vorticity gradient

\[
\frac{\partial E_i^p}{\partial t} = -\frac{\partial}{\partial t} \int \overline{h_1^{-2} v Q_1^p (\eta_1^2 / 2)} dA
\]

Here, \( \eta_1 \) is the horizontal displacement of particles in the upper layer and \( Q_1 = (1 + v_1) / \overline{h_1} \) is the basic state upper layer potential vorticity. In the present case of uniform upper layer potential vorticity, \( Q_1 = 0 \), so the wave energy
does not change with time. In fact, since no unstable wave is present initially, $E_T^\theta$ is zero for all time. The instabilities found in the present case have zero net energy flow between the basic state and disturbance fields (this fact is verified numerically below).

An alternative expression for the time rate of change of the wave energy is (see the appendix in part 1 for details)

$$
\frac{\partial E_T^\theta}{\partial t} = - \int \left[ \frac{\partial w}{\partial x} u_1 v_1 + \frac{\partial v}{\partial x} u_1 u_1 - \frac{\partial w}{\partial x} (u_2 + v_2) v_1 - \frac{\partial v}{\partial x} u_1 \zeta_2 - \frac{\partial w}{\partial x} u_1 \zeta_2 \right] dA
$$

(3)

The first two terms represent horizontal Reynolds stresses, and the third term symbolizes the vertical Reynolds stress. The fourth term represents the process of baroclinic instability or the flux of water in the cross-front direction which tends to flatten out the warped interface. The final term in the integrand can be placed on the left-hand side of (3) by relating it to the change in time of the displacement of the surface front (see Barth [1987] for details). It is useful to analyze these energy conversion terms to help elucidate the primary physical mechanism for the instability.

The other interpretation of instability involves the exchange of disturbance energies between two resonating stable modes. The unstable mode grows with time keeping its disturbance (mean plus wave) energy equal to zero. The flow of energy in this case is between the two stable partners, depending on the basic state configuration, the unstable mode may gain from, lose to, or leave unchanged (as in the present case for a uniform potential vorticity flow) the energy of the basic state flow. Further discussion of these ideas can be found in the work of Hayashi and Young [1987]. Since both interpretations yield useful information, both are considered here.

The contours of $\zeta_1$ and $\zeta_2$ provide phase information for the unstable mode. The sense of phase tilt in an unstable disturbance is useful in interpreting the direction and mechanism of energy transfer in the system [Pedlosky, 1986]. The signature of an unstable wave extracting energy from the basic state potential energy field via baroclinic instability is upper layer perturbation fields lagging those below. This sense of phase tilt, such that the perturbation is “leaning” against the basic state vertical shear, is apparent in Figure 3. Therefore it may be anticipated that the term on the right-hand side of (3) representing this process is positive. The direction of the transfer of kinetic energy via the horizontal Reynolds stress is more difficult to determine by inspection of Figure 3a. The first horizontal Reynolds stress term on the right-hand side of (3) will be positive (conversion of basic state kinetic energy to the perturbation) if the phase of the perturbation is again leaning against the basic state horizontal shear. This is clearly the case in Figure 3a, but the presence of the additional term in (3) proportional to the horizontal Reynolds stress, but with opposite sign, complicates the interpretation. In fact, for this uniform basic state potential vorticity case the wave energy should be zero from (3), so if the baroclinic conversion is from the basic state flow to the perturbation, then the conversion of kinetic energy via the Reynolds stress must be of the opposite sign. A numerical evaluation of the energy balance presented below does indeed confirm this result.

The solutions obtained numerically can be used to calculate the terms in the energy statements (1)–(3) explicitly. The terms in the definition of wave energy (equation (1)) are plotted as a function of cross-front distance in Figure 4. The solid curve is the positive definite sum of the kinetic energy in each layer and the potential energy. The long-dashed curve is the correlation between $v_1$ and interface displacement which results in a negative contribution to the wave energy. This results because the unstable disturbance decreases the total speed ($v_1 > 0$) where it deepens the upper layer ($\zeta_2 < 0$; see part 1, Figure 3). The sum of these positive and negative contributions to the wave energy is plotted as the short-dashed curve in Figure 4. Its integral over the domain of the fluid vanishes, confirming the result that unstable waves on basic state flows with uniform upper layer potential vorticity have “zero wave energy.”

The energy conversion terms in (3) due to the action of Reynolds stresses and baroclinic instability are plotted as a function of $x$ in Figure 5. The values in the legend result from integrating the various terms over the domain of the fluid. Only the Reynolds stress terms are plotted in Figure 5a. The first term in the integrand of (3) representing the horizontal Reynolds stress $-\mu_1 \zeta_2 v_1$ is positive, as was inferred from the examination of the phase tilt in the unstable disturbance. The second term $-\mu_1 \zeta_2 v_1$ is negative and large with the result that the net horizontal Reynolds stress acts to convert energy from the perturbation to the mean flow. The vertical Reynolds stress term $-\mu_1 \zeta_2 (u_1 + v_1) v_1$ is positive and small, and the final term in the integrand of (3), which earlier had been related to the deflection of the surface front, is large and negative. The sum of the Reynolds stress terms displayed in Figure 5a is plotted as the dashed curve in Figure 5b. The solid curve in Figure 5b represents the baroclinic energy conversion term, whose integral over the domain of the fluid is positive and exactly balances the integral of the sum of the other terms. The integrand in (3) can be related to the mean energy $E_T^\theta$ by using the fact that $\partial (E_T^\theta + E_T^\zeta)/\partial t = 0$ (see part 1, section 3)
that $\frac{\partial E_{\gamma}}{\partial t} = \int (\text{energy conversion terms}) \, dA$. The fact that the sum of the terms in the integrand is zero confirms that the mean energy, as well as the wave energy (as illustrated in Figure 4), is zero for this basic state flow with uniform upper layer potential vorticity. The ratio of the magnitude of the baroclinic conversion term to the magnitude of the Reynolds stress terms (neglecting the contribution from the term, $\overline{\nu_2 u_1} \zeta_2$, which is related to the deflection of the surface front) is $\sim 2.6$, which indicates that the energy conversion process, although mixed, is primarily baroclinic.

An expression for the change in time of the wave momentum is (see part 1, section 3, for details)

$$\frac{\partial M_2^\gamma}{\partial t} = -\frac{\partial}{\partial t} \int \left[ \frac{h_2^2}{h_1^2} Q_{1s}(\eta_2^2/2) + \frac{h_2^2}{h_2^2} Q_{2s}(\eta_2^2/2) \right] \, dA \tag{4}$$

where

$$M_2^\gamma = \int (w_2 - v_1) \zeta_2 \, dA \tag{5}$$

Here $\eta_2$ is the horizontal displacement of particles in the lower layer and $Q_{1s} = 1/h_2$ is the basic state lower layer potential vorticity. Even though $Q_{1s} = 0$, the gradient of the lower layer basic state potential vorticity ($Q_{2s}$) is nonzero and negative. The calculated value for the wave momentum ($M_2^\gamma$) from (5) is positive, while that of the mean momentum ($M_2^\gamma$), from (4) and the fact that $\frac{\partial M_2^\gamma}{\partial t} = -\frac{\partial M_2^\gamma}{\partial t}$, is negative. These two exactly balance, so that the total disturbance (mean plus wave) momentum is zero as it must be for an unstable disturbance in this system.

The structures of the two stable waves at $l = 2.0$, which for $l \leq l_c$ had been resonating to produce the unstable mode, are presented in Figure 6. The velocities are again scaled for each wave such that the vector beneath the legend represents one unit. Both stable waves are frontally trapped and have highly ageostrophic velocity fields. The wave with the larger phase speed ($c_r = -0.2722$) (Figure 6a) is primarily trapped in the upper layer, while the slower wave ($c_r = -0.0578$) (Figure 6b) has comparable velocity magnitudes in each layer (a more “barotropic” mode). The two waves have the same sign of phase propagation but oppositely directed group velocities (defined by $c_g = \partial \sigma/\partial \mu$). The wave with $\sigma_r = -0.5444$ has a negative group velocity and thus propagates energy toward $-\mu$. The opposite is true for the other stable wave. An explicit calculation of the terms in the energy statements (1)–(5) can also be performed on the two stable modes whose alliance results in the unstable mode analyzed earlier. The wave with $c_r = -0.2722$ has negative disturbance energy, while the wave with $c_r = -0.0578$ has positive disturbance energy. When these two waves have the same wave number and phase speed, they can exchange disturbance energies to form an unstable mode which has zero disturbance energy. Thus the unstable mode can grow through a transfer of disturbance energy from one of its partners to the other.

Fig. 5. Energy conversion terms defined in (3) versus cross-front distance: (a) horizontal Reynolds stresses $-h_2 \overline{\nu_2 u_1} v_1$ (solid curve) and $-h_2 \overline{u_1 u_1} v_1$ (long-dashed curve), vertical Reynolds stress $-h_2 \overline{u_1 (u_1 + v_1)} v_1$ (short-dashed curve), and $\overline{\nu_2 u_1} \zeta_2$ (dot-dashed curve); (b) conversion of energy via baroclinic instability $-\overline{u_1} \zeta_2$ (solid curve) and the sum of the terms plotted in Figure 5a (dashed curve).
The preceding paragraphs detail the existence of unstable waves on a uniform basic state potential vorticity flow, confirm the energy statements reviewed here (equations (1)–(5)) and presented more thoroughly in part I, section 3, and discuss the mechanisms for their existence. The variety of mean flow profiles and layer depths associated with coastal upwelling fronts found in nature motivates an investigation of the effect of changing the model parameters. The first parameter to be varied is the distance from the front to the coastal wall ($x_f$). The maximum growth rate ($\sigma_{\text{max}}$) is plotted versus $|x_f|$ in units of $R$ in Figure 7. The wave number of the fastest growing wave ($l_m = 1.05$) does not change appreciably (less than 1%) as $|x_f|$ is varied. Moving the front closer to the coastal wall results in a decrease of the maximum growth rate. This decrease is $\sim 34\%$ for the surface front immediately adjacent ($x_f = 0$) to the coastal wall. Note that when the surface front is immediately adjacent to the coastal wall the system is still unstable. The growth rate asymptotes to a constant value as $|x_f|$ increases with the choice $|x_f| = 2.0 R$ being similar to the no wall case considered by Killworth et al. [1984]. In fact, comparison of their numerical results (see their Figure 4) are in excellent agreement with the results presented in Figure 2a. This agreement also supports the validity of the geostrophic momentum approximation [Hoshina, 1975], which was used in part I to simplify the numerical solution technique, for this choice of basic state flows. In summary, maximum growth rates decrease as the surface front is moved closer than $\sim 2 - 3 R$ from the coastal wall.

Since the stability theorem derived in part I establishes the stability of a uniform basic state potential vorticity flow with an infinitely deep lower layer (a reduced gravity model) and the results of this section show the existence of a strong instability (large growth rate) for equal layer depths, a dramatic dependence of the growth curve on lower layer depth may be expected. Growth rate versus alongfront wave number with $x_f = -2.0 R$ for various values of $r$, the nondimensional total depth as defined above, is plotted in Figure 8. Deep lower layers decrease the range of unstable wave numbers, shift the fastest growing wave to longer wavelengths (smaller $l$), and dramatically decrease the growth rate of the fastest growing wave. These results are in agreement with the shallow water equation results of Killworth et al. [1984]. Since the source of energy for the unstable wave is primarily baroclinic, the decrease in growth rate can be understood in light of the decreasing overall vertical shear of the basic state flow. For continuous stratification and velocity the growth rate due to baroclinic instability for a given wave number is limited by the magnitude of the vertical shear [Pedlosky, 1986]. An analogous statement for layer flows may be established using a finite difference form of the vertical shear where $\Delta z$ is the depth of the water.
column even though, in reality, the vertical shear in the system is concentrated at the interface. As the lower layer depth increases, the overall vertical shear decreases and the baroclinic energy conversion process becomes weaker. The absolute stability of the flow with an infinitely deep lower layer occurs essentially because the lower layer becomes so massive that interaction with the upper layer is removed. In other words, the baroclinic conversion of energy is absent, and thus the flow becomes stable. The numerical solution presented in Figure 8 is correctly converging to this limit.

The discussion in part 1, section 3, illustrated the importance of the shape of the interface profile (as denoted by $\alpha$) to the stability of a coastal upwelling front. It was also shown that the signs of the wave and mean energy components of the unstable disturbance depend on the signs of the basic state potential vorticity gradients (characterized by $\alpha$). The effect on the stability properties of the system of varying $\alpha$ is now addressed. Growth rate versus alongfront wave number for three values of $\alpha$ are presented in Figure 9a. The wave number of the fastest growing wave decreases slightly ($\sim 5\%$ for a $20\%$ increase in $\alpha$) as the interface becomes "steeper" (i.e., cross-front $e$-folding scale less than a Rossby radius of deformation). A more significant effect of increasing $\alpha$ is the increase in growth rate of the fastest growing wave. Growth rate ($\sigma_1$) and the real part of the frequency ($\sigma_r$) as a function of $\alpha$ for $l = 1.0$ are plotted in Figure 9b. The growth rate and the real part of the frequency increase smoothly as $\alpha$ passes through 1.0. Since the growth mechanism is via baroclinic instability, this increase in growth rate is related to the increase in overall vertical shear between the two layers.

Numerical evaluation of the definitions of energy for the
most unstable wave shows that for $\alpha < 1.0$ (cross-front e-folding scale greater than the Rossby radius of deformation) the instability has $E_2^3 > 0$ and $E_1^3 < 0$ and thus can be labelled “positive wave energy” instability. In other words, for $\alpha < 1.0$ the traditional form of instability is recovered. This may be rationalized by realizing that in the limit as $\alpha$ becomes small the horizontal shear in the system decreases and (ignoring the $O(1)$ change in layer depth) traditional quasi-geostrophic dynamics become more applicable. For $\alpha > 1.0$, the unstable mode has $E_2^3 < 0$ and $E_1^3 > 0$. This “negative wave energy” instability is in the same class as those recently analyzed by Marimont and Risa [1984].

The growth rate curve for an interface profile with $\alpha = 1.2$ shows another mode of instability at high wave numbers (Figure 9). This unstable mode was not found for any value of $\alpha < 1.0$. The growth rate for this mode increases without bound for increasing wave number. As $l$ becomes larger, the trapping scale (which goes $\sim l^{-1}$) becomes shorter and shorter. Observations of fronts often show a great deal of mixing near the surface front and inclusion of this process (following Carrive [1984]) may influence (e.g., completely quench) this high wave number mode trapped closely to the surface front.

Inspection of this high wave number mode reveals it to be an instability whose energy conversion is from the kinetic energy of the basic state to the disturbance. The structure of $\zeta_1$, the surface displacement, shows a rapid phase change across the point where the phase speed of the wave equals the mean flow speed ($\zeta_\infty = \zeta_1$). This “critical layer” behavior is similar to the results of Killworth [1983], who found unstable modes in a reduced gravity model only when $\alpha$ was greater than 1.0. Comparing the real part of the eigenvalue ($\sigma_r$) as a function of $l$ for $l$ greater than $\sim 1.5$ calculated here with Killworth’s [1983] result shows them to be in excellent agreement. However, his results indicate that the growth rate of this mode decreases with increasing $l$ for $l$ greater than $\sim 1.5$.

Clearly, the geostrophic momentum approximation, which was used in part 1 to simplify the numerical solution technique, fails to produce accurate results at these high wave numbers for flows with values of the interface parameter $(\alpha)$ greater than 1.0. This unreliable behavior at high wave numbers is a symptom of the arbitrary truncation of the governing equations as discussed in part 1, section 2 (see also McWilliams and Gent [1980]). Accurate results for this parameter range may be obtained only through consideration of the full shallow water equations. A more thorough discussion of this topic can be found in part 1 and in the work of Barth [1987].

In conclusion, a two-layer, flat-bottom model of a coastal upwelling front with uniform basic state potential vorticity in the upper layer is unstable. The unstable mode propagates phase in the same direction as the mean flow (toward $-y$) and converts potential energy of the basic state flow to perturbation energy via baroclinic instability. Simultaneously, the perturbation transfers kinetic energy back to the basic state flow so as to leave the basic state flow unaltered. This results in both the mean and wave energies of the disturbance being identically zero. The unstable mode still grows with time and energy transfer occurs between two resonating wave partners. The proximity of the front to a coastal barrier affects the growth rate of the unstable mode. The growth rate of the fastest growing wave decreases the closer the front is to the wall once that distance is less than $\sim 2 - 3 \ R$. The depth of the lower layer dramatically influences the properties of the unstable mode. Shallow lower layers increase the range of unstable wave numbers and increase the growth rates at all wave numbers. Increasing the vertical shear in the system (greater $\alpha$) increases the growth rate of the unstable mode. For $\alpha > 1.0$ an unstable mode with negative wave energy is found, while for $\alpha < 1.0$ the unstable mode has positive wave energy. In addition, for $\alpha > 1.0$ there exists a one-layer instability whose phase speed is correctly found using the geostrophic momentum approximation but whose growth rate is severely miscalculated. Caution should thus be used when modelling flows in this large wave number range (short alongfront scales) using the geostrophic momentum approximation.

### 2.2 Inviscid, Sloping Bottom Model

In this section the effect of sloping bottom topography on the stability of a coastal upwelling front is investigated. All of the regions where coastal upwelling fronts are known to occur contain sloping continental shelves (the region between the coastal barrier and the continental slope). The bottom slope can be classified as strong, in the sense that the change in water depth across the shelf is comparable to the total depth. As mentioned in section 1, these strong bottom slopes are not allowed in quasi-geostrophic theory, where the fractional change in depth must be of the same order as the Rossby number which is assumed small. However, the model used here, which employs the geostrophic momentum approximation applied to the shallow water equations, permits strongly sloping bottom topographies. The two-layer models of instability by Orlanski [1969], who used the shallow water equations, and by Mechoos and Sinton [1981], who used the quasi-geostrophic equations, both show that bottom topography slope in the same sense as the slope of the interface stabilizes the system (i.e., reduces the growth rates of the unstable waves). The model of Orlanski [1969] did not consider the stability of a flow with a surface front in the vicinity of a strongly sloping bottom. In the preceding section the strong destabilizing effect of a shallow lower layer was demonstrated. Since coastal upwelling fronts usually form in shallow water over a sloping bottom, it is of interest to find out the net effect of these two opposing influences on the stability of the front.

In this section the effect of a linearly sloping bottom on the stability of a front with uniform basic state potential vorticity in the upper layer is discussed. The total depth takes the form $H = H_0 + z$. As discussed in section 1, the sloping bottom joins a flat-bottom region at a distance several Rossby radii offshore of the surface front. In section 3, the influence of bottom topography which is an arbitrary function of $z$ and specifically chosen to model actual continental shelf profiles is presented. It is desirable to keep the change in the depth below the surface front as small as possible while varying the bottom slope because most of the energy conversion occurs within $\sim 1$ Rossby radius of the surface front (Figures 4 and 5). For this reason, the sloping bottom was pivoted about a point fixed directly beneath the surface front (see Figure 10b). The values of the bottom slope given in Figure 10b are obtained by multiplying the actual physical bottom slope by $R/H_1$. This makes one unit of vertical distance equal to one unit of horizontal distance.
Before detailed results are presented, the effect of a sloping bottom on the stability of the front can be anticipated by applying the general stability criteria developed in part 1. To satisfy part of the criteria which echo the familiar quasi-geostrophic requirements for stability, the gradient of the basic state potential vorticity must not change sign anywhere (i.e., within either layer or between layers). Since the upper layer potential vorticity is uniform in this case, the change of sign must occur in the lower layer. If the lower layer is continuously thickening as \( z = -\infty \), then the flow would be stable as judged by quasi-geostrophic criteria. For the linearly sloping bottom of the present case, the bottom slope must be greater than the maximum slope of the interface for continuous thickening of the lower layer. For the exponentially sloping interface, the maximum interfacial slope occurs at the surface front and is equal to \(-1\) for the uniform potential vorticity front considered here. Therefore the quasi-geostrophic criteria are satisfied for \( s \leq -1.0 \). However, the additional ageostrophic stability constraint (see part 1, section 3) is still violated for some range of \( s \leq z_1 \), so the possibility for unstable modes still exists.

Growth rate (\( \sigma_\nu \)) versus alongfront wave number (\( l \)) for models with various values of \( s \) are displayed in Figure 10a. The solid curve for a flat-bottom model is the same as that in Figure 8 for a total nondimensional depth of 3 (\( r = 3 \)). The long-dashed curve is for \( s = -0.125 \), and the short-dashed curve is for \( s = -0.5 \). The presence of a sloping bottom decreases the growth rate of the unstable wave at all wave numbers. The high wave number cutoff (\( l_c \)) becomes smaller for increasing slope. The sloping bottom also shifts the wave number of the fastest growing wave (\( l_o \)) to longer wavelengths. Note that even though the depth of the lower layer beneath the interface is increasing as the bottom slope steepens, the fractional decrease in the growth rate of the fastest growing wave (\( \sigma_{o_\nu} \)) is more than expected from a depth increase alone as illustrated in Figure 8. The slope itself is stabilizing the system. The shift of the unstable modes to smaller wave numbers occurs because the wave’s cross-front velocity in the lower layer is restricted by the sloping bottom topography. In other words, the wave motion in the lower layer is forced to be more alongshore. In order to maintain a circulation in the lower layer roughly equivalent to that found in the flat-bottom case (Figure 3b), the closed cell, which has been compressed in the cross-shelf direction due to sloping bottom topography, elongates in the alongshore direction. This leads to a larger alongfront wavelength or a smaller value of the alongfront wave number. For a bottom slope of \( s = -1.0 \), no unstable modes were found. The possibility of unstable waves even when \( s \leq -1.0 \) was anticipated above by an examination of the general stability criteria. Violation of the necessary condition for instability is, in this case, not enough to insure the existence of an unstable mode.

Properties of the unstable modes which exist in the presence of sloping bottom topography are examined by focusing on the case \( s = -0.5 \). Phase speed (\( c_\nu \)) versus alongfront wave number (\( l \)) is plotted in Figure 11, where the unstable phase speed is denoted by a solid curve and a number of stable modes are plotted as dashed curves. Except for the two modes which resonate for \( l < 0.8 \) to produce the unstable mode, the stable vorticity modes with \( c_\nu < 0 \) are not plotted. The phase speed of the unstable mode is less negative than in the flat-bottom case (compare Figure 2b). This results because the sloping bottom induces phase propagation to \(+y\), which is the same direction as topographic vorticity waves propagate. In fact, the dashed curves for \( c_\nu > 0 \) are stable vorticity modes propagating with shallow water on their right. Only the first four stable modes with \( c_\nu > 0 \) have been plotted. There exist many more stable waves with \( c_\nu > 0 \) whose phase speeds decrease with increasing mode number. Among the stable vorticity modes are familiar continental shelf waves and, for this case where the interface intersects the surface, the complimentary mode edge waves of Bane [1980].
The structure of the unstable mode for an alongfront wave number near the fastest growing wave is presented in Figure 12. The upper layer structure is similar to that for the flat-bottom mode (Figure 3a) and, as discussed above, the circulation in the lower layer is elongated in the alongshore direction. The small growth rate for this mode is consistent with the upper and lower layer perturbations being more nearly in phase, indicating a less effective conversion of potential energy from the basic state flow.

The transition from an unstable front over a flat bottom to a front completely stabilized by a linearly sloping bottom \((s \leq -1.0)\) is illustrated in Figure 13. The phase speed \((c_r)\) is plotted versus alongfront wave number \((l)\) for both stable (dashed curves) and unstable (solid curves) modes for a range of the values of \(s\). For the flat-bottom case \((s = 0)\), Figure 13a) there are no stable modes with \(c_r > 0\). A number of stable modes exists for \(c_r < 0\), but only the modes with the three largest values of \(|c_r|\) are plotted. The front is unstable over the entire range of \(l\) plotted. With a small amount of bottom slope \((s = -0.06)\), Figure 13b) stable modes with \(c_r > 0\) are introduced. As in Figure 13b, only the first four stable modes with \(c_r > 0\) are plotted. The phase speeds of the stable modes with \(c_r < 0\) have become less negative. The front is still unstable for wave numbers in the range \(0.1 \leq l \leq 1.2\), but now the unstable mode propagates to \(+y\) for \(l\) less than \(\sim 0.37\). The fact that these unstable modes with \(c_r\) outside the range of the mean flow speeds (which is negative for all values of \(x\)) exist can be rationalized by realizing that the topographic slope in this study is analogous to the effect of \(\beta\) (the north-south gradient of the Coriolis parameter) in a flat-bottom model. In a quasi-geostrophic model with \(\beta\), a semicircle theorem can be established which allows unstable waves with phase speeds outside the range of the mean flow velocities [Pedlosky, 1986]. Analogous possibilities are allowed in this study due to the linear bottom slope.

For a greater value of \(|s|\) (Figures 13c–13e), the phase speeds of the stable modes with \(c_r > 0\) become greater. The phase speeds of the stable modes with \(c_r < 0\) become more positive and appear as a thick line near \(c_r = 0\) in Figures 13d and 13e. Unstable waves exist for a smaller and smaller range of \(l\) as \(|s|\) increases. Finally, for \(s = -1.0\) (Figure 13f), no unstable modes are found, and there exists only one stable mode with \(c_r < 0\).

Another useful experiment involves keeping the bottom slope constant and increasing the lower layer depth. Intuitively, the stability of a surface front over a deep lower layer might not be expected to depend on the bottom topography. However, results (not presented here) indicate that the stabilizing effect of a sloping bottom is not diminished by a deep lower layer. This result is consistent with the stability theorem (part 1), since no matter how deep the lower layer is, there still exists a change in sign of the lower layer basic state potential vorticity gradient for \(s > -1.0\). In the real ocean, surface-intensified features often appear to be independent of the bottom topography. The model discussed here has a homogeneous lower layer, so that bottom effects are transmitted to the entire water column. In the real, continuously stratified, ocean, the underlying density field may negate the effect of the bottom topography on the stability of the surface front. This omission in the simple model investigated here is discussed further in section 4.

The above results indicate that a large absolute value of the bottom slope parameter \(s\) may stabilize a coastal upwelling front. However, alongfront variability is commonly observed on coastal upwelling fronts over slope topography (i.e., the front is not stable). The resolution to this contradiction is that modelling the continental shelf as a constant linear slope is not very realistic. Upwelling regions of the world’s oceans contain quite variable bottom topographies with both steeply and gently sloping sections. Since in their formation process, coastal upwelling fronts migrate offshore [e.g., de Smeed and Richman, 1984], they pass over regions of varying bottom slope. Therefore the front’s stability is influenced by the local bottom slope near (within \(\sim 1\) Rossby radius) the surface front. Including a realistic bottom topography provides a range of the values of the bottom slope parameter \(s\) which may contain small enough absolute values to allow for the possibility of unstable waves. A model with bottom topography which is an arbitrary function of \(x\), the cross-shore direction, is used in section 3 to verify the existence of unstable waves on realistic coastal upwelling fronts.

Since instability is a frontolitic effect, the results so far suggest that in the absence of other forces (e.g., friction, wind stress) coastal upwelling fronts would most likely be found where the bottom topography is steep and where the water depth below the front is deep. This hypothesis may not hold in reality because wind stress and frictional effects are rarely absent. For example, the time scale of the wind forcing in upwelling regions is too short (\(\sim 3–5\) days) to allow the unforced evolution proposed above to occur. However,
the frontolytic effects of instability may have a greater influence on the equilibrium position of more permanent fronts in the coastal ocean such as the shelf break front in the Middle Atlantic Bight off New England.

2.3 Linearized Bottom Friction

The results discussed previously have all been for an inviscid model. Since coastal upwelling fronts form in regions of shallow water, bottom friction will certainly be present. In addition, viscous effects may be present at the sharp density interface between layers [Simpson and James, 1986]. While interfacial friction may be important in the formation and evolution of a coastal upwelling front, its inclusion precludes simple analytic solutions for even the basic state interface profile. Therefore in the model investigated here, damping is provided to the flow by linearized bottom friction. The representation of bottom friction in the governing equations and its effect on the numerical solution technique are discussed in the appendix.

The primary model used to study the effect of bottom friction is the uniform potential vorticity flat-bottom model discussed in section 2.1. The offshore constant lower layer depth is chosen to be twice the upper layer depth \( (r = 3) \). The value of the friction coefficient, \( \lambda_s \), (related to the dimensional version which has units of velocity by \( \lambda_s = fH_1/\lambda_s \)) chosen for the study was 0.2, which corresponds for an upper layer depth of 20 (50) m and mid-latitudes \( (f \sim 10^{-4}) \) to a dimensional value of 0.04 (0.10) cm s\(^{-1}\). Typical values for the continental shelf appear to be 0.015–0.08 cm s\(^{-1}\) [Brink et al., 1987]. The model friction parameter of 0.04 cm s\(^{-1}\) for \( H_1 = 20 \) m lies within this range, while the value of 0.10 cm s\(^{-1}\) for \( H_1 = 50 \) m is probably an overestimate.

To determine what effect friction might have on the stability of the coastal upwelling front, it is useful to compare the frictional time scale (defined as the ratio of water depth to the friction parameter) to the e-folding time scales of the growing, inviscid, unstable waves. For the fastest growing wave on a uniform potential vorticity front with the offshore lower layer depth equal to twice the upper layer depth \( (r = 3) \), growth curve displayed in Figure 8), the dimensional e-folding growth time scale is \( \sim 1.8 \) days. If the total depth is used, the dimensional barotropic frictional time scale \( (3H_1/\lambda_s) \) is \( \sim 1.7 \) days. Since these time scales are comparable, bottom friction may play a major role in the stability of a coastal upwelling front. In fact, previous quasi-geostrophic instability results including damping show that friction may destabilize a system by introducing new modes of instability which are absent in the inviscid cases [Holopainen, 1961]. However, since the unstable wave motion is concentrated in the upper layer for the present model (Figures 3 and 12), the effect of bottom friction may be reduced due to the insulating presence of the lower layer. Following Allen [1984], the effective friction parameter due to stratification is obtained by multiplying \( \lambda_s \) by \( (1/2)(H_1/H_2) \). Since \( (H_1/H_2) = 1/2 \), the effective fric-
Fig. 14. Growth rate versus alongfront wave number for an inviscid (dashed curve) model front and a model with linearized bottom friction ($\lambda = 0.2$, solid curve). Both models have a flat bottom ($z = 0$, $r = 3$) and uniform upper layer basic state potential vorticity ($\alpha = 1.0$).

The present of dissipation, disturbance energy is no longer conserved. For the unstable alliance to occur, the wave partner must have negative disturbance (given by the sign of $E^\nu_\i$) energy so it can lose disturbance energy to dissipation. The high wave number mode found here is an example of this process which is described further by Cairns [1979].

The high wave number mode found here in the presence of friction can be compared to the high wave number mode found in the inviscid model with $\alpha > 1.0$ (see section 2.1 and Killworth [1983]). Both modes have negative wave energy. For the inviscid model with $\alpha > 1.0$, the vorticity source for the perturbations is the nonzero basic state potential vorticity gradient in the upper layer. In the viscous case, bottom friction provides the vorticity source for the perturbations. In either case, the high wave number modes are trapped closely to the front and have small growth rates. As mentioned in section 2.1, these modes may not be present in nature not only because they grow slowly, but also because cross-frontal mixing near the surface front may quench them completely.

Results for a model with a flat bottom and $\alpha \neq 1.0$ are qualitatively similar to those discussed here for $\alpha = 1.0$. Specifically, the growth rate of the fastest growing wave for all $\alpha$ is decreased and a slowly growing, high wave number mode exists in the presence of friction for $\alpha < 1.0$. The high wave number mode for $\alpha = 1.2$, whose growth rate is mis-calculated using the geostrophic momentum approximation (see section 2.1), is essentially unaffected by the presence of bottom friction.

The primary effects of bottom friction on the unstable modes discussed above are to decrease the fastest growing mode's growth rate and increase its phase speed in the direction of the mean flow. When the e-folding growth time scale of the wave and the frictional damping time scale are comparable, friction may destabilize the system [Holopainen, 1961]. To illustrate this effect, the model was run with a uniform basic state potential vorticity flow in the upper layer ($\alpha = 1.0$) and a sloping bottom such that $z = -1.0$. As shown in section 2.2, the inviscid version of this model is absolutely stable. When bottom friction is added to the system, a weakly growing ($\sigma_m \approx 6.0 \times 10^{-3}$ for a model with $h_0(x) = 3.0$; compare to growth rates in Figure 14) unstable mode appears. This destabilization by friction in the presence of topography is analogous to the quasi-geostrophic models with the $\beta$ effect and bottom friction [Holopainen, 1961].

The destabilization of the flow through the introduction of viscosity is a counterruptive result. This effect can be rationalized in several ways. One simple explanation is that bottom friction breaks the connection between the interior flow and the bottom slope. That is, the strong constraint for flow along isobaths is broken by the presence of bottom friction. A second explanation, as mentioned earlier, involves an alliance between dissipation and a stable wave with negative disturbance energy. The stable wave propagating to $-y$ in the inviscid model with $z = -1.0$ (see Figure 13f) has negative disturbance energy. This mode grows as it loses disturbance energy to dissipation. A final interpretation, couched in terms of traditional instability theory, is as follows. Friction introduces a phase shift in the disturbance. This phase shift allows the wave to release energy from the basic state flow. It can be shown that the gain in energy is
larger than the loss of energy to dissipation so the disturbance amplifies.

In conclusion, bottom friction, which is known to be an important process in the coastal ocean, is not expected to quench completely the unstable waves present on a coastal upwelling front. This is mainly due to the fact that the motion of the unstable wave is concentrated in the upper layer and is thus insulated from the damping effect of bottom friction by the presence of the lower layer [Allen, 1984]. However, bottom friction does provide a sink of energy to the system, so a decrease in the growth rate of the fastest growing mode can be expected. Finally, an example of destabilization by bottom friction in the presence of sloping bottom topography is given.

3. Comparison with Observations

The results from the simple stability analysis presented in the previous sections show that unstable waves exist on model coastal upwelling fronts. The alongfront wavelength, e-folding time, and propagation speed of the fastest growing wave depends on various properties of the basic state flow and model geometry. Observations of upwelling fronts in many regions of the world's coastal oceans show alongfront variability in the offshore position of the surface front which often appear wavelike and can extend alongshore over many repeated wavelengths. Observations also yield evidence for temporal growth in the size of these alongfront disturbances. In this section, observations of upwelling fronts from three regions of the world's coastal oceans are examined for evidence of unstable frontal waves. The discussion presented here is not intended to be a comprehensive worldwide survey of frontal variability, nor is it meant to be a detailed examination of the physical dynamics of coastal upwelling. Rather, case studies from selected upwelling regions are analyzed concentrating on the properties and evolution of wavelike perturbations in the coastal upwelling front. These regions are off the coasts of Oregon, southwestern Africa, and Nova Scotia. To realistically model these regions, the stability model is modified to include bottom topography which is an arbitrary function of z, the cross-shelf coordinate.

3.1 Oregon

The coastal ocean off Oregon is an area of active upwelling during the summer when winds become predominantly "upwelling-favorable" (blowing alongshore with the coast to the left in the northern hemisphere) [e.g., Huyer, 1983]. An example of a coastal upwelling front in this region is presented in Figure 15. Two time periods existed in summer 1973 during which almost daily aircraft sea surface temperature (SST) maps are available [O'Brien et al., 1974]. The first period is July 12-15 after the peak of a strong (wind stress greater than 4 dyn cm\(^{-2}\)) upwelling-favorable wind event (Figure 16). SST maps on July 12, 14, and 15 are shown in Figure 17. On July 12 (Figure 17a), active upwelling is indicated by the presence of cold water near the coast. The transition between inshore, cold, upwelled water and the warmer, offshore waters occupies a region at least 35 km wide. Note the tendency for the surface isotherms to run nearly parallel to the coast. However, they are not absolutely parallel to the coast or the bottom topography. An alongfront meander exists in the surface isotherms (e.g., the 49° contour).

Fig. 15. A density section near 45°15'N off the coast of Oregon during July 1973. The region of compressed isopycnals intersecting the surface ~ 10 km offshore and continuing seaward at ~ 20 m depth is the coastal upwelling front. From Curtin [1979].

On July 14 the wind stress has decreased (Figure 16) as the upwelling-favorable wind event concludes. The SST map (Figure 17b) shows a sharp, pronounced surface front ~ 20 km offshore. An alongfront meander pattern, which appears wavelike and extends alongshore over approximately two wavelengths, is clearly evident. The northern part of the disturbance has a larger wavelength than the southern part. A quantitative estimate of the wave properties is presented below. Tracing isotherm position (e.g., using the 49°
Fig. 17. Sea surface temperature maps off the coast of Oregon derived from aircraft radiometer measurements taken on (a) July 12, (b) July 14, and (c) July 15, 1973. Temperature contours (heavy solid curves) are in degrees Fahrenheit, and isobaths are in fathoms (1 fathom = 1.83 m). The flight path is shown as a dotted line. From O'Brien et al. [1974].
isotherm) from the SST maps of July 12 to July 14 shows that the alongfront meander has grown (larger cross-front peak-to-peak amplitude). Note that following the motion of individual surface isotherms between maps may be complicated by processes such as mixed layer deepening. However, in this case and in the others discussed in this section, individual isotherm motion is indicative of the evolution of the SST pattern as a whole and therefore provides a reliable source for estimating wave properties. Growth in time of a meander pattern does not alone justify the interpretation of the phenomenon in terms of an instability process. For instance, stable wind-driven motions may change in amplitude as the forcing changes in time. However, growth in time and propagation of the meander pattern in the same direction as the basic state flow (the latter is established below for the case studies from each of the three geographic locations) do support the idea that the observed wavelike perturbations result from an instability process.

The wavelike pattern evident in Figure 17b does not have a symmetric sinusoidal shape. There is clear evidence that the surface front's position moves more quickly from offshore to onshore than from onshore to offshore. In other words, the wave is steeper downstream of regions where the upper layer is shallow. This type of pattern is consistent with the alongfront advection of layer height implied by the negative contribution of the term \(-\delta H \zeta_2\) to the wave energy discussed in section 2.1. This contribution implies that regions where the upper layer is shallow travel more quickly downstream than regions with thick upper layers (see part 1, Figure 3). A plan view of this pattern as obtained from a conceptual model of wave steepening is displayed in Figure 18. Two comments can be made on the wave steepening effect. First, wave steepening is a nonlinear effect, and the unstable wave motions modelled here are strictly linear. However, wave steepening is consistent with the predicted linear solution and does appear in observations of alongfront variability. Second, the wave steepening observed here is in the opposite sense to that predicted for stable waves on surface fronts over sloping bottom topography. Gill and Schumann [1979] show that for a surface front over a linearly sloping bottom, only stable waves with poleward speeds less than the mean flow speed exist. They further note that for this supercritical flow, the inshore part of the wave moves more quickly to \(-y\) than the offshore part. This leads to wave steepening downstream of the region where the upper layer is deep. Therefore the observations presented here are more consistent with unstable frontal waves than with stable topographic waves in the presence of a surface front.

Compression and rarefaction of surface isotherms within the frontal zone is apparent in Figure 17b. This pattern may be due to the presence of a shallow bottom and a coastal barrier on one side of the front. On the inshore side of the front, the bottom and coast restrict the cross-shelf movement of the surface isotherms which leads to their compression. On the offshore side, the peaks grow without restriction, so no compression occurs and the isotherms appear spread apart compared to the inshore peaks. Evidence for this topographic effect is apparent in the widely spaced isotherms at the offshore peak near \(\sim 45^\circ15'N\) and the tightly packed isotherms at the inshore peak just to the south. While the above discussion may explain some of the observed compression and rarefaction of surface isotherms, the structure of the frontal zone is clearly more complicated. A stability model with better resolution of the frontal zone is needed to understand the observed surface temperature patterns fully.

The SST map from July 15 (Figure 17c) exhibits a sharp surface front in the north (45°10'N) connected with a weaker temperature gradient region to the south. The vertical section of density presented in Figure 15 was taken across the sharp front in the north at \(\sim 45^\circ15'N\). There still exists alongshore variability in the position of the upwelling front, but the amplitude of the disturbance appears less than in the July 14 SST map. Furthermore, tracing the change in time of isotherm position (e.g., using the 49°isotherm) shows that the entire meander pattern has propagated to the south. This propagation speed is quantified below. The sharp front in the northern part of the July 15 map is presumed to have propagated into the study region from the north.

The next available SST map (July 16, not shown) exhibits a less organized pattern and contains several eddies of \(\sim 30\ km\) length in the alongshore direction. It is difficult to identify a surface front continuously connected in the alongshore direction.

In order to compare the observations described qualitatively above with the stability model predictions, a quantitative estimate of the meander properties must be made. To establish an alongfront wavelength, individual SST maps are used to measure the alongfront distance between peaks of the wave. As noted above, the wavelength is different between the northern and southern parts of the disturbance. This difference may be due to an alongshore change in the shelf geometry or basic state flow, a feature which is absent in the simple stability model presented in the previous sections. An estimate of the range of alongfront wavelengths is obtained by measuring peak-to-peak distances from all three SST maps. Individual values obtained in this manner are contained in Table 1. The range of alongfront wavelengths is 32–52 km.

Fig. 18. Modeled example of surface isotherm pattern due to wave steepening effect. The coastal barrier lies at the top of the figure. At a constant offshore distance seaward of the surface front (e.g., \(-1.2\)), the upper layer is deep when the front is closest to the coast and shallow for the front farthest from the coast.
TABLE 1. Estimates of the Properties of Observed Alongfront Variability off the Coast of Oregon

<table>
<thead>
<tr>
<th>Date</th>
<th>Wavelength km</th>
<th>Phase Speed cm s(^{-1})</th>
<th>Cross-Front Peak-to-Peak Amplitude km</th>
</tr>
</thead>
<tbody>
<tr>
<td>July 12</td>
<td>32, 47</td>
<td>-9, 11, 15, 12</td>
<td>6, 7</td>
</tr>
<tr>
<td>July 14</td>
<td>47, 52, 34</td>
<td>-33, -36, -20, -12</td>
<td>9, 10, 14</td>
</tr>
<tr>
<td>July 15</td>
<td>39, 34</td>
<td>-9</td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>41</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

While estimating alongshore phase propagation for the meander patterns is difficult, an attempt is made to establish at least its direction and magnitude. Two methods are used to determine the alongshore propagation speed. Changes in the alongshore position of the wave peaks between July 12 and July 14 and between July 14 and July 15 are averaged to obtain an average propagation speed of \(\sim 9\) cm s\(^{-1}\) to the south. The individual estimates used in this average are contained in Table 1. The second method involves measuring the distance between the intersection of individual surface isotherms with the 50-fathom isobath from one SST map to the next. This method yields an average propagation speed of \(\sim 8\) cm s\(^{-1}\) toward the south.

An estimate of growth rate for this frontal wave is found by examining the change in time of the cross-front peak-to-peak wave amplitude. This is most easily done between the July 12 and July 14 SST maps, with the result (individual values in the average are contained in Table 1) \(\sigma f^{-1} \sim 4\) days.

Before comparing observed frontal wave properties for this geographic location to those obtained from the stability model, a second time period is examined. SST maps from July 22 and 23 are displayed in Figure 19. The winds during this period (Figure 16) vary daily between a 1 dyn cm\(^{-2}\) stress in an upwelling-favorable direction to little or no wind. A surface temperature front is evident in Figure 19a separating cold, inshore water from the warmer water offshore. As during the previous time period (July 12–15), a wavelike meander exists on the surface front. No SST information is available in the 3 days before the July 22 map, so the previous time history of the meander pattern is unknown. The wavelength of this feature is comparable to that of the July 12–15 meander, with the same tendency for a longer length scale in the northern part of the study region. The properties of this wave are quantified below. A continuously connected alongshore front is less obvious in the SST map from July 23 (Figure 19b). In fact, the wavelike pattern of July 22 seems to have amplified and perhaps broken into closed or nearly closed eddies. Note the offshore eruption of the 56°C isotherm at the northern end of the study region, the deepening of the trough just to the south and the strengthening of the cold eddy at \(\sim 44^\circ 50'\)N. This sequence of SST images offers tentative evidence for disruption of the alongshore flow field by the formation of detached eddies.

The same techniques as used for the first time period are used to quantify the properties of this wavelike perturbation. The individual estimates used are contained in Table 2. The range of alongshore wavelengths is 31–60 km.

An alongshore phase speed of \(\sim 14\) cm s\(^{-1}\) to the south is obtained from noting the change in time of the alongshore position of wave peaks. From examining the movement of individual isotherms along the 50-fathom isobath, a value of \(\sim 11\) cm s\(^{-1}\) is obtained. Growth rate is again a difficult property to estimate, but the e-folding time during this
TABLE 2. Estimates of the Properties of Observed Alongfront Variability off the Coast of Oregon

<table>
<thead>
<tr>
<th>Date</th>
<th>Wavelength km</th>
<th>Phase Speed cm s⁻¹</th>
<th>Cross-Front Peak-to-Peak Amplitude km</th>
</tr>
</thead>
<tbody>
<tr>
<td>July 22</td>
<td>60, 37</td>
<td>10, 21, 9, 0</td>
<td>7, 8, 4, 6</td>
</tr>
<tr>
<td>July 23</td>
<td>31, 31, 47</td>
<td>28, 33</td>
<td>15, 9</td>
</tr>
<tr>
<td>July 24</td>
<td>47</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>42</td>
<td>−14</td>
<td></td>
</tr>
</tbody>
</table>

The results from these two case studies of alongshore meanders in the coastal upwelling front off Oregon provide estimates of wave properties which can be compared with predictions from the simple two-layer stability model. Since the properties of the growing waves predicted by the stability model are greatly affected by the choice of layer depths and bottom topography, it is necessary to have an accurate estimate of these values for the study region off Oregon. A constant upper layer depth far offshore of the surface front can be estimated from examining the vertical section of density displayed in Figure 15. The value of $H_1$ chosen is 20 m, with a density defect of $\delta = 0.002$. With these values, the internal Rossby radius of deformation ($R$) is 6.2 km and the long internal gravity wave phase speed $(g^*H_1)^{1/2}$ is 63 cm s⁻¹. From Figure 15 it is difficult to determine the interface profile due to the presence of many small-scale features. In the stability model, the interface is modelled as an exponential with an adjustable e-folding scale. It was decided to use $\alpha = 1.0$ as a reasonable first guess, rather than estimate an e-folding scale from observations. As described in section 2.1, increasing $\alpha$ makes the interface rise more steeply to the surface, increases the maximum velocity at the surface front and, as a result, increases the growth rate of the fastest growing wave. However, the wave number of the fastest growing mode is not greatly affected by a change in $\alpha$ (see Figure 9a). With $\alpha = 1.0$, the maximum upper layer mean flow velocity is 63 cm s⁻¹ to the south at the surface front. This value is consistent with the geostrophic velocity estimates of Curtin [1979], who found velocities of $\sim 40-50$ cm s⁻¹ at 20 m depth near the surface front. For the above reasons, the model interface parameter is chosen as $\alpha = 1.0$.

The cross-shelf bottom topography varies alongshore in the study region (Figure 17), and an effect which is not included in the simple stability model. The cross-shelf bottom topography used in the stability model is displayed in Figure 20a. Other cross-shelf sections in the region have more steeply or more gently sloping continental shelves, but the topography displayed in Figure 20a is fairly representative. As discussed in section 2.2, an increase (decrease) in the bottom slope decreases (increases) the growth rate and the alongfront wave number of the fastest growing wave. The scaled bottom slope parameter $s$ as a function of cross-shelf distance is displayed in Figure 20b. This value is obtained by multiplying the actual physical bottom slope by $R/H_1$. As noted in section 2.2, with $\alpha = 1.0$ a value of $s \lesssim -1.0$ beneath the surface front stabilizes the front. Since, the bottom topography in Figure 20 contains a range of values of $s$, the offshore location of the surface front may determine its stability properties. Another point to keep in mind is that deeper lower layers decrease the growth rate and alongfront wave number of the fastest growing wave (Figure 8). Therefore it is anticipated that the fastest growing mode may occur when the surface front lies over a weakly sloping bottom in shallow water.

The inviscid stability model is run with the above parameter choices and for a range of values of the distance between the surface front and the coastal $x_{f*}$ estimated from the SST maps in Figure 17. Model-predicted and observed wave properties are presented in Table 3. No unstable waves are present.

Fig. 20. (a) Bottom topography off the coast of Oregon used in the stability model. (b) Bottom slope parameter $s$ for the cross-shelf topography in Figure 20a. The parameter $s$ is related to the actual physical bottom slope $s_*$ by $s = (R/H_1)s_*$. Time period appears to be $\sim 2$ days, somewhat shorter than observed in the previous sequence of images.

The inviscid stability model is run with the above parameter choices and for a range of values of the distance between the surface front and the coastal $x_{f*}$ estimated from the SST maps in Figure 17. Model-predicted and observed wave properties are presented in Table 3. No unstable waves are present.

TABLE 3. Comparison Between Observed and Modelled Properties of Alongshore Meanders in the Coastal Upwelling Front off Oregon

<table>
<thead>
<tr>
<th></th>
<th>Wavelength km</th>
<th>Phase Speed (to the south) cm s⁻¹</th>
<th>e-folding time days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observations</td>
<td>31–60</td>
<td>8–14</td>
<td>2–4</td>
</tr>
<tr>
<td>Model 1</td>
<td>52–56</td>
<td>1–2</td>
<td>5–8</td>
</tr>
<tr>
<td>Model 2</td>
<td>37–43</td>
<td>3–6</td>
<td>7–11</td>
</tr>
</tbody>
</table>

Model 1 input parameters are $H_1 = 20$ m, $\delta = 0.002$, $\alpha = 1.0$, $-22$ km $< x_{f*} < -15$ km, $\lambda = 0.0$ and bottom topography displayed in Figure 20 (see text for details). Model 2 input parameters are as in model 1, but with bottom friction ($\lambda = 0.04$ cm s⁻¹).
found for a model with the surface front closer than ~12 km to the coast. The largest value of the fastest growing wave's growth rate occurs for \( x_f \approx -19 \) km. This mode has an alongfront wavelength of ~52 km. The maximum growth rates for \( x_f = -16 \) km and \( x_f = -22 \) km are less, but they occur at a wave number only slightly less than that found for \( x_f = -19 \) km. The alongfront wavelengths predicted by the two-layer stability model are within the range of the observed values. However, the model underpredicts both the phase speed toward the south and the growth rate. These properties may be modified by adjusting the model input parameters. For example, increasing \( \alpha \) (i.e., making the model front rise more sharply to the surface) will increase the phase speed toward \(-y\) and decrease the e-folding time due to the increased vertical shear. The vertical section of density in Figure 15 provides an indication that the pycnocline may rise to the surface sharply. Another possible explanation for the underprediction of phase speed and growth rate is the lack of stratification beneath the model front since underlying stratification may isolate the upper water column from the influence of the sloping bottom. However, given the reasonable input parameters used here, the simple two-layer stability model predicts properties of the fastest growing wave which are fairly close to the observed values. Therefore the existence of growing alongfront meanders on coastal upwelling fronts in this region may be interpreted in terms of an internal instability process.

One final note concerns the effect of linearized bottom friction on the unstable wave. An estimate of the frictional time scale for the model front off Oregon using \( \lambda = 0.04 \) cm s\(^{-1}\) is 18 days. Since this time scale is long compared to the growth rates predicted by the inviscid model (5-8 days), bottom friction is expected to decrease the fastest growing wave's growth rate and increase its phase speed to \(-y\) (see section 2.3). Running the viscous stability model with a bottom friction parameter of \( \lambda = 0.04 \) cm s\(^{-1}\) (Table 3) shifts the predicted phase speed closer to the observed values, but contributes further to the model underprediction of growth rate.

### 3.2 Southwestern Africa

The coastal ocean off the southwestern tip of Africa is an area of active upwelling [Bang, 1978] during the austral summer when winds from the southeast drive an upwelling circulation which creates coastal upwelling fronts. Upwelling has been observed from south of Cape Point (34°S) to well north of Hondeklip Bay (30°S) (Figure 21). A number of regions of locally intense upwelling exist within this area [Taumont-Clark, 1985]. Though these upwelling structures are three-dimensional, there are also areas where a fairly two-dimensional front forms alongshore. One such area is west of Cape Town between Cape Point and Cape Columbine. A number of observational studies have taken place in this region, including the collection of a large set of aircraft-derived SST maps [Taumont-Clark, 1982]. The purpose of this section is to describe the growth of an alongfront meander on the coastal upwelling front as portrayed in a series of daily SST maps and to evaluate the model against this information.

During January 17–28, 1980, the wind consistently blew in an upwelling-favorable direction [Jury, 1984]. Within this period, the wind stress is fairly constant at \( \approx 1 \) dyn cm\(^{-2}\) from January 21 to January 27. On January 22 an aircraft SST survey was done from south of Cape Town to north of Cape Columbine (Figure 22c). The alongshore spacing of the sampling grid (not shown) is ~ 25 km. Cold water is found adjacent to the coast, with an upwelling tongue extending northward from the Cape Peninsula (34°S). Offshore of this tongue is a region of compressed surface isotherms running alongshore over most of the survey region. This coastal upwelling front, centered on ~ 50 km offshore, is the region of interest here. On January 22 there appears to be an alongfront meander with two peaks away from the coast at ~ 33°15' S and ~ 34°15' S. The distance between these two peaks is ~ 125 km. On January 23 (Figure 22b), most of the SST features are still present from the previous day. The offshore surface front seems to have sharpened in the southern half of the study region. The alongfront meander is still present and appears to have shifted slightly northward (e.g., see the 15° isotherm). At the southern end of the survey region another peak in the alongfront meander is now evident.

The SST map from January 24 (Figure 22c) shows the amplification of the alongfront meander. All three peaks away from the coast increase in amplitude, whereas the inshore peaks do not. Again, there is some propagation of the meander pattern to the northwest. The final SST map in this case study is from January 25 (Figure 22d). Growth in the alongfront meander is evident, but the pattern has not shifted significantly in the alongfront direction. Note the sharpening of the surface front at the inshore peaks, especially near 34°45' S. This apparently occurs because the cross-shelf movement of the inshore meander peaks is restricted by the topography resulting in a compression of the surface isotherms. There is some evidence that the wave is steeper downstream of regions where the upper layer is shallow (e.g., the peak near 34°15' S). This pattern is consistent with the wave motion in the unstable mode found from the simple stability model as discussed in section 3.1 above.

As in the previously discussed study off Oregon, quantitative estimates can be made for the properties of the ob-
served waves described qualitatively above. Estimates from individual maps are given in Table 4, and a summary of observed values is contained in Table 5. To obtain predictions for the same properties, the simple stability model is run with bottom topography from a cross-shelf section originating near 33°35'5S (Figure 23a). The bottom slope parameter $s$ is shown as a function of cross-shelf distance in Figure 23b.

No subsurface density information is available during the time the SST maps discussed above were collected. To estimate $H_1$ and $\delta$ for input into the model, vertical sections through similar upwelling fronts during the same season but for a different year are used [Bang, 1973]. The values used are $H_1 = 70$ m and $\delta = 0.0015$ which lead to a Rossby radius of 12.4 km and a long internal gravity wave phase speed of
TABLE 4. Estimates of the Properties of Observed Alongfront Variability off the Southwestern Coast of Africa

<table>
<thead>
<tr>
<th>Date</th>
<th>Wavelength (km)</th>
<th>Phase Speed (cm s⁻¹)</th>
<th>Cross-Front Peak-to-Peak Amplitude (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan. 22</td>
<td>125</td>
<td>5, 16, 6, 6, 9, 12</td>
<td>4, 9, 10</td>
</tr>
<tr>
<td>Jan. 23</td>
<td>150, 81</td>
<td>54, 9, 34, 21</td>
<td>11, 2, 14, 16, 4</td>
</tr>
<tr>
<td>Jan. 24</td>
<td>132, 95</td>
<td>-29, 2, -26, 4</td>
<td>6, 2, 10, 5, 7</td>
</tr>
<tr>
<td>Jan. 25</td>
<td>125, 90</td>
<td>-8, 11</td>
<td>12, 8, 11, 11, 20, 16</td>
</tr>
<tr>
<td>Average</td>
<td>114</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

101 cm s⁻¹. Again, the interface parameter is α = 1.0 as a reasonable first guess. From the aircraft SST maps (Figure 22) the front is ~ 40–50 km offshore which, given the bottom topography displayed in Figure 23, places the front in 150–170 m of water. The properties of the fastest growing waves using the above model inputs and varying the offshore position of the surface front are presented in the second row of Table 5. While the predicted alongfront scale compares reasonably well with the observed values, the phase speed and growth rate are too slow. Both these values could be increased by increasing the steepness of the interface profile (i.e., making the overall vertical shear near the surface front larger). As in the previous case studies, the presence of underlying stratification, an effect omitted in the present stability model, may isolate the upwelling front from the stabilizing influence of a sloping bottom.

To assess the influence of bottom friction on the stability properties of the system, a comparison between the e-folding time of the fastest growing wave and the frictional damping time scale can be made. Using a value of λₚ = 0.04 cm s⁻¹, the frictional time scale is estimated as ~ 14 days. This time scale is comparable to the e-folding times found above, so the possibility of destabilization by friction, as discussed in section 2.3, is possible. Results from a viscous model using the above value of λₚ appear in the final row of Table 5. Linearized bottom friction does improve the model estimates of phase speed and growth rate, but the model still underpredicts the growth rate.

One difference between this case study and the previous observations off Oregon is the presence of a fairly constant upwelling-favorable wind stress during the observation period. In the previous case study, upwelling-favorable winds occur in events separated by periods of weak or downwelling-favorable winds. The simple stability model developed here does not include wind stress and should therefore be compared only with observations obtained during these periods of weak winds between upwelling events. However, as discussed in part 1, the stability model may apply equally well to basic state flows which exist as a result of a steady state balance between wind forcing and dissipation. In other words, these two processes would then not enter the stability analysis other than through their effects on the basic state flow field. This case study off southwestern Africa suggests that the simple model may be used to analyze the stability of a coastal upwelling front in the presence of wind stress. The introduction of time-dependence in the basic state flow field from the action of wind stress may, however, affect the stability of the front. This point is discussed further in section 4.

3.3 Nova Scotia

Recently, Petrie et al. [1987] presented observational evidence for unstable waves on a coastal upwelling front which formed off the eastern coast of Nova Scotia. Their descriptive results are summarized here. During the month of July 1984, a weak but persistent alongshore wind blew in an upwelling-favorable direction. In response, cold saline water was upwelled at the coast forming an offshore surface front. An alongfront meandering with wavellike character-
istics was observed to grow on this surface front. A time series of maps showing the position of the 14°C isotherm as subjectively sketched from Petrie et al.'s [1987] color-enhanced satellite SST images is presented in Figure 24. The 14°C isotherm separates colder water inshore from warmer waters offshore. As best seen in the map from July 25 (Figure 24c), three large meanders in the coastal upwelling front extend offshore along the southeastern coast of Nova Scotia. There is also evidence for two more, smaller-amplitude, meanders along the front to the northeast. These perturbations are fairly evenly spaced alongshore and have a wavelength in the range of 50–75 km [Petrie et al., 1987].

The growth in time of these meanders is evident in Figure 24. The earliest map (July 14, Figure 24a) shows a region of cold water along the Nova Scotian coast and around into the Bay of Fundy. The alongfront perturbations in the surface front have begun to form at this time. The three southernmost wavelike meanders grow to rather large amplitude by the July 21 SST map (Figure 24b). The map from July 25 (Figure 24c) shows the fully developed meanders extending up to ~75 km away from the initial surface front. In the following week (July 31, Figure 24d) the area of upwelled water decreases in size and warms. At the same time the alongfront meanders decrease in size.

From this series of SST maps the existence of growing frontal perturbations is clearly established. By superimposing successive images, Petrie et al. [1987] find a phase propagation to the northeast in the range of ~2.0 cm s\(^{-1}\). These features grow quite rapidly, as is evident in Figure 24. Petrie et al. [1987], using additional SST images not presented here, estimate the e-folding time to be on the order of several days. A summary of the observed unstable wave properties is contained in Table 6.

As in the previous case studies, estimates of layer depths and bottom topography need to be made in order to predict wave properties from the two-layer stability model. No detailed subsurface density sections across the upwelling front are available from this time period, but two hydrographic stations were occupied in late July. One station is located just off the central southeastern coast of Nova Scotia within the band of cold water, and the other is located ~75 km offshore, outside the cold band. The offshore station allows an estimate of 20 m for \(H_f\) with a density defect of \(\delta = 0.003\). This gives a long internal gravity wave speed of 76 cm s\(^{-1}\) and an internal Rossby radius of deformation equal to 7.4 km.

The bottom topography off the eastern coast of Nova Scotia is rather complicated. The water depth increases rapidly nearshore, reaching a value of 100 m within 10–12 km of the coast. Offshore of this the bottom slopes gently with a nearly constant linear value of ~1 × 10\(^{-5}\) [Petrie et al., 1987] to a deep trough ~80 km offshore. Finally, an offshore bank gives way to a steep continental slope beginning at ~180 km offshore. Since the SST maps indicate that the surface front lies over the region of approximately constant bottom slope, the two-layer stability model was run with a linearly sloping bottom next to a vertical coastal barrier as in section 2.3. The final necessary model input is the offshore distance to the surface front, which is estimated from the SST maps to be ~15 km. This places the surface front in water of ~110 m depth. Since no details of the cross-front subsurface density structure are available, the interface parameter is chosen as \(\alpha = 1.0\).

The properties of the fastest growing wave from the model appear in row 2 of Table 6. The predicted wavelength and phase speed are within the range of observed variability. However, as in the previous case studies, the model under-predicts the growth rate of the fastest growing mode. Again, the model input parameters may be varied. For example, a 20% increase in the vertical shear (obtained by increasing \(\alpha\) to 1.2) results in no change in alongfront wave number but increases the alongfront phase propagation to 2 cm s\(^{-1}\) to the northeast and decreases the e-folding time to 7 days. Another possibility, as mentioned in conjunction with the example from Oregon, is that underlying stratification may isolate the upwelling front from the stabilizing effect of the sloping bottom.

Bottom friction may also be included in this model. Since the water depth is large beneath the surface front, the frictional time scale is expected to be long. With a value of \(\lambda = 0.04\) cm s\(^{-1}\), this scale is ~30 days. Clearly, the fastest growing wave in the inviscid model grows on a time

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**TABLE 6. Comparison Between Observed and Modelled Properties of Alongshore Meanders in the Coastal Upwelling Front off Nova Scotia**

<table>
<thead>
<tr>
<th>Wavelength (km)</th>
<th>Phase Speed (to the northeast) cm s(^{-1})</th>
<th>e-folding time days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observations</td>
<td>50–75</td>
<td>0–2</td>
</tr>
<tr>
<td>Model 1</td>
<td>74</td>
<td>1</td>
</tr>
<tr>
<td>Model 2</td>
<td>60</td>
<td>4</td>
</tr>
</tbody>
</table>

Model 1 input parameters are \(H_f = 20\) m, \(\delta = 0.003\), \(\alpha = 1.0\), \(x_f = -15\) km, \(\lambda = 0.0\) and linearly sloping bottom topography (see text for details). Model 2 input parameters are as in model 1, but with bottom friction (\(\lambda = 0.04\) cm s\(^{-1}\)).

---

Fig. 24. A time series of maps from the region southeast of Nova Scotia showing the location of the 14°C isotherm separating colder water inshore (gray shading) from warmer water offshore. The heavy solid curve is the 200-m isobath, which represents the approximate location of the shelf break in this area. After Petrie et al. [1987].
scale shorter than this. The results from running the viscous stability model with $\lambda = 0.04 \text{ cm s}^{-1}$ are contained in the final row of Table 6.

Petrie et al. [1987] use a three-layer quasi-geostrophic model to study this phenomenon, since they find that a two-layer quasi-geostrophic model without horizontal shear is completely stabilized by the strong bottom slope. The presence of a deep third layer with zero mean flow effectively isolates the two upper flowing layers from the stabilizing influence of a sloping bottom. Even though the large interface displacements associated with coastal upwelling fronts make the quasi-geostrophic models formally invalid, they obtain properties of the fastest growing wave which are in the range of observed values. Their predicted alongfront phase speed of 4–5 cm s$^{-1}$ to the northeast is larger than that expected from the observations. They suggest that adding a 3 cm s$^{-1}$ depth-independent flow to the southwest in the system may bring this predicted value in line with observations. The authors also apply a constant depth two-layer shallow water equation model (Killworth et al. [1984] and section 2.1 in this study) to the problem. Again, predicted wave properties are within the range of observed values except for an excessive northeastward phase propagation.

The results presented here are from a model which is more physically realistic than either of the above two models. Large interface displacements and strong bottom slopes are included. Two points are noteworthy. First, sloping bottom topography representative of the continental shelf off the eastern coast of Nova Scotia does not stabilize the two-layer frontal model as it did the two-layer quasi-geostrophic model with uniform mean flow in each layer. Second, as discussed in section 2.2, a sloping bottom induces a shift in the phase speed of the unstable waves. The sense of this shift is to add a component in the direction that topographic vorticity waves propagate (to the southwest off the eastern coast of Nova Scotia). Since the modelled unstable frontal waves propagate in the downstream direction (toward the northeast off the eastern coast of Nova Scotia during the period of active upwelling described here), a sloping bottom will decrease the magnitude of the phase speed. The observed small values of alongfront phase speed to the northeast for unstable waves in this area are likely due to this effect.

4. DISCUSSION

Several of the more serious omissions made in simplifying the stability model and the role of unstable frontal waves in coastal circulation are discussed here. Representing the water column beneath the interface as a single homogeneous layer omits the possible influence of the underlying stratification apparent in Figure 15. Since the motion of the unstable mode discussed in section 2 is strongest in the upper layer, the density structure beneath the interface is not expected to affect greatly the wave's properties. However, as discussed in sections 2.3 and 3, the underlying stratification may stabilize the upper water column from the influence of the bottom. As a result, a sloping bottom may not stabilize the front to the degree it does with a homogeneous lower layer. In addition, the damping effect of bottom friction may be reduced. The result of both of these effects is that the front may be more unstable than predicted here.

The lack of cross-frontal mixing and/or interfacial friction is another weakness of the model. These processes are important in the formation and evolution of a coastal upwelling front [de Sroeder and Richman, 1984]. The removal of energy from the upper layer via interfacial friction and the degradation of the sharp interface and surface front by mixing would most likely lead to a decrease in the growth rate of the unstable waves predicted by this simple model. However, observations of frontal variability (see section 3) suggest that these processes do not completely quench the unstable waves.

Linear instability theory is restricted to describing the small-amplitude behavior of unstable waves. A nonlinear calculation is required to describe the evolution of the instability once it reaches finite amplitude. However, reasonable agreement between the scales of observed alongfront (finite amplitude) variability and the model predictions suggests that the linear theory presented here provides results which are likely to hold up at later times as the flow becomes increasingly nonlinear. Sometimes, waves which grow exponentially with time at small-amplitude stop growing or enter limit cycle oscillations upon reaching finite amplitude [Pedlosky, 1966]. As discussed in section 3, the finite amplitude meanders still appear to be growing, which suggests that the predicted growth from the linear theory may continue as the unstable waves become increasingly nonlinear. In fact, continual growth of the meanders would eventually produce detached eddies for which there is some observational evidence (see Figure 19b). To verify these ideas and to study the evolution of the meanders and their influence on the mean alongshore flow, a nonlinear calculation is required.

Another possible shortcoming of the model developed here is the lack of wind stress. As described in section 1, an alongshore wind stress is essential in the formation of a coastal upwelling front. The stability model presented here takes the fully developed, wind-formed front as its starting point. Some observational evidence suggests that coastal upwelling fronts are more stable (i.e., two-dimensional) in the presence of a strong, steady wind [Curtin, 1979]. After the cessation, reversal or weakening of the wind event the front is observed to meander. Other observations, such as those presented here from off the southwestern coast of Africa, show that alongfront meanders can exist in the presence of a steady wind. In addition, the coastal upwelling front in the laboratory models of Narimousa and Mazurzky [1987] is unstable for all values of the applied surface stress. The observational and experimental evidence does not establish clearly the effect of wind stress on the stability of a coastal upwelling front.

The stability properties of the front may be affected by changes in the basic state flow field which arise due to wind forcing. Specifically, a steady wind stress will drive an upwelling circulation which depends on time. The alongshore flow speed will increase with time and the surface front will migrate offshore. Time-dependence in the basic state flow will also result from the action of a time-dependent wind stress. The stability model presented here employs a fixed basic state flow. A wind stress may also affect the stability of a coastal upwelling front through the interaction of the wind-driven Ekman flow with the front. To study these possibilities, a model which resolves Ekman layer flows and allows time-dependence is required.

A final major weakness of the model, as alluded to in section 3, is the lack of alongshore variability in the bottom
topography and/or shoreline configuration. In section 2, the profound influence of the local lower layer depth and the bottom slope on the properties of the unstable waves was presented. Therefore the stability of the upwelling front may be different as the alongshore jet flows between regions of different topography. The theory presented here is valid for wave motions with alongshore scales much smaller than the alongshore scale of the topographic variability. This may hold true in some regions of active upwelling for which the model predictions will be valid. However, other observations have suggested that alongshore topographic variations are important to coastal circulation. Studies off the northern California shelf have demonstrated that regions near capes and points are areas of enhanced upwelling [Kelly, 1985] and vigorous cross-shelf velocities [Davis, 1985a]. A three-dimensional model is needed to address the influence of alongshore topographic variability on the stability of a coastal upwelling front.

One potentially fruitful way to study the finite-amplitude behavior of these unstable waves in the presence of a time-dependent wind stress and/or alongshore topographic variations is to use a fully nonlinear, three-dimensional model of a coastal upwelling front. Experiments varying the wind stress and/or the alongshore bottom topography and/or the coastline configuration would help to resolve the questions raised above.

The unstable frontal waves discussed in this study may have an important role in the circulation associated with a coastal upwelling front. As mentioned in the introduction, the waves may grow to finite amplitude and redirect the alongshore flow in the cross-shelf direction. This redirection by large-amplitude unstable frontal waves is only one possible explanation for the existence of strong, narrow, offshore flows (squirts and jets). While other mechanisms involving variations in alongshore bottom topography, and/or coastline configuration and/or wind stress have been proposed [Hartwig and Brink, 1985], the instability of the flow field associated with the coastal upwelling front remains a likely candidate.

Unstable waves on coastal upwelling fronts may also contribute to the cross-shelf eddy flux of water properties. In their study of the heat budget of the coast of Oregon in 1973, Bryden et al. [1980] found that the cross-shore eddy heat flux is an important process. They note that the eddy heat flux is directed down the mean horizontal temperature gradient and thus removes potential energy from the mean circulation, a behavior consistent with the unstable waves discussed here. The presence of a coastal upwelling front during the time of their study (Figures 15, 17, and 19) lends further support to the possibility that the eddy flux came from the type of instability modelled here. Since Bryden et al. [1980] only analyzed data from one current meter mooring they were unable to estimate an alongshore scale for the eddy motions. Recently, Lentz [1987], using data from three alongshore moorings deployed as part of the Coastal Ocean Dynamics Experiment (CODE), found alongshore variability in the cross-shelf eddy heat flux with alongshore scales of less than 56 km. This alongshore scale provides an upper bound to compare to the size of the finite amplitude eddies which may evolve from the unstable frontal waves modelled here. That these eddy motions may be due to instability associated with the coastal upwelling front is further corroborated by the presence of the front near the moorings during the analysis period of the study [Lentz, 1987]. Davis [1985a], using drifter data from CODE, also found vigorous eddy variability on scales of 40 km or less. None of the above studies establish a lower bound for the alongshore scale of the eddy variability. To summarize, the above studies indicate the importance of cross-shelf eddy heat flux to the heat budget for an area of active upwelling. The cross-shelf eddy flux of other water properties (e.g., nutrients) is also of practical importance. The unstable waves discussed here provide a mechanism for the existence of these eddy fluxes.

5. Conclusions

Observed alongfront variability on a coastal upwelling front can be explained in terms of an instability process. The existence of unstable waves on a model coastal upwelling front, as suggested by the stability theorem developed in part 1, is confirmed by solving directly the linearized equations of motion. An unstable mode is described which propagates phase in the same direction of the mean alongfront flow and which converts potential energy of the basic state flow to perturbation energy via baroclinic instability. The unstable wave motion is frontally trapped and is concentrated in the upper layer. The upper layer flow is ageostrophic, while the lower layer motion is weaker and more geostrophic and consists of closed cells beneath the surface front. The predicted properties (i.e., alongfront wavelength, phase speed, and growth rate) of the fastest growing waves from the simple two-layer stability model are in reasonable agreement with the observed scales of variability from three regions of active upwelling in the world's coastal ocean.

Changes in the basic state flow parameters and the model geometry affect the stability properties of the system. Moving the front closer than ~2 Rossby radii to the coastal wall results in a moderate decrease (by a maximum of ~34% when the surface front lies immediately adjacent to the coastal barrier) in the growth rate of the fastest growing wave. Increasing the overall vertical shear of the basic state flow, by either decreasing the lower layer depth or increasing the steepness of the interface, results in an increase in the growth rate of the fastest growing wave. The effect of changing the lower layer depth is particularly dramatic, leading to complete stability for flows with uniform upper layer potential vorticity over an infinitely deep lower layer.

Including a bottom linearly sloping in the same sense as the interface results in a decrease of the growth rate and an increase in the alongfront wavelengths of the unstable waves in the system. Complete quenching of the unstable frontal waves by steep bottom slopes is not likely to occur for realistic coastal upwelling fronts for two reasons. First, bottom slope magnitudes vary across the continental shelf, so that at some location the bottom may be sloping gently enough to allow instability. This result is verified by comparing predictions from a model with arbitrary cross-shelf bottom topography to observed alongfront variability from three regions of active upwelling.

Second, the presence of any density structure (not included in this study) beneath the interface may insulate the upper part of the water column from the stabilizing effect of a sloping bottom. Bottom friction, which is important in coastal circulation, is included and results in a decrease (by as much as 40% in one particular case) in the growth rates of the unstable waves by extracting energy from the system. However, since the
unstable wave motion is strongest in the upper layer, bottom friction does not completely stabilize the upwelling front. An example of destabilization by bottom friction in the presence of sloping bottom topography (analogous to destabilization by friction in a quasi-geostrophic model which includes the beta effect) is given.

This simple instability model describes a likely mechanism for the formation of eddies in the alongshore currents which accompany coastal upwelling. To investigate further the role of unstable frontal waves in the complicated eddy and jet current structures observed over continental shelf and slope regions, the model may be extended to include finite amplitude effects and alongshore topographic variations.

### Appendix

#### A1. Solution Technique

Since the coefficients of the linearized perturbation equations (equations (8) and (9) in part I) depend on $x$ only, solutions are sought which are periodic in $y$. The normal mode form appropriate to this study (using perturbation sea surface elevation as an example) is

$$\zeta (x, y, t) = R \left[ \zeta_1 (x) \exp (iy \omega t) \right]$$  \hspace{1cm} (A1)

Here $l$ is the (real) alongfront wave number and both frequency $\omega$ and cross-front amplitude function $\zeta_1 (x)$ are complex. $R$ denotes the real part of the expression in square brackets. Since $\omega = \sigma + i \omega$ is complex, unstable solutions with $\omega > 0$ grow exponentially with time. With $\sigma > 0$, the initial disturbance grows until the nonlinear terms neglected in the linearized equations become large. When this occurs, the linear theory presented here is no longer adequate to describe the evolution of the unstable waves. Nevertheless, linear theory successfully describes the initial instability process and provide details of the small-amplitude behavior of the unstable waves.

The solution procedure proceeds as follows (a detailed description is given by Barth [1987]). Substituting (A1) and similar expressions for $\zeta_T$ and the layer velocities into the linearized perturbation equations yields (dropping hats)

$$\left( \sigma - \nu \right) \zeta_1 + v_1 = \zeta_{1x}$$  \hspace{1cm} (A2a)

$$-i(\sigma - \nu)\zeta_1 + (1 + \nu_2)u_1 = -i\nu_1 \zeta_1$$  \hspace{1cm} (A2b)

$$-i(\sigma - \nu)\left( \zeta_1 - \zeta_T \right) + (u_1 \overline{h}_1)_x + i\nu_1 \overline{h}_1 = 0$$  \hspace{1cm} (A2c)

for the upper layer and

$$\sigma \zeta_T + v_2 = \zeta_{2x}$$  \hspace{1cm} (A3a)

$$-i\sigma \zeta_{2x} + u_2 = -i\nu_2 \zeta_T$$  \hspace{1cm} (A3b)

$$-i\nu_2 \zeta_T + (u_2 \overline{h}_2)_x + i\nu_2 \overline{h}_2 = 0$$  \hspace{1cm} (A3c)

for the lower layer. Next, the first two equations in (A2) and (A3) are used to solve for the layer velocities which are then substituted into the continuity equations (A2c) and (A3c) to get a single equation for each layer. The result for the upper layer is

$$-i \zeta_{1x} + \left[ \frac{\nu - \frac{\nu_x \overline{h}_1}{1 + \nu_2}}{(1 + \nu_x)} \right] \zeta_{1x}$$

$$- (1 + \nu_1 \nu_2)(1 + \nu_x) \zeta_1 + (1 + \nu_2) \zeta_T$$

$$= \frac{i \sigma}{\overline{h}_1} \left( \frac{\nu_x \overline{h}_1}{1 + \nu_x} \right) \zeta_{1x}$$

$$- \left[ (1 + \nu_1 \nu_2)(1 + \nu_x) - 1 + \frac{\nu_{xx} \overline{h}_1}{\overline{h}(1 + \nu_x)} \right] \zeta_1$$

$$+ \left( 1 + \nu_2 \right) \zeta_T$$

(A4a)

The lower layer equation becomes

$$\zeta_{Txx} + \frac{\nu_{xx}}{\overline{h}_2} \zeta_{Tx} - \left[ \frac{1}{\overline{h}_2} + \nu_2 \right] \zeta_T + \frac{1}{\overline{h}_2} \zeta_1 = \frac{ih_{2x}}{\sigma \overline{h}_2} \zeta_T$$

(A4b)

The complex frequency $\sigma$ is the eigenvalue for this problem and appears only linearly in these two coupled equations. If the geostrophic momentum approximation is not made, the eigenvalue appears nonlinearly in the shallow water equation equivalent of (A4).

Note that the upper layer equation (A4a) is singular at several points within the domain of the fluid. Singularities exist where $\overline{h}_1 = 0$ (at the front, $x = x_f$), where the horizontal shear of the mean flow exactly balances the Coriolis parameter $(1 + \nu_x = 0)$ and at a "critical layer" where the phase speed of the wave equals the mean flow speed ($\sigma = \overline{\nu}$ or $c = \overline{\nu}$ with $c = \sigma \overline{\nu}$ being the wave phase speed). These singularities make the analytic solution of (A4) (and the reduced gravity version of (A4a)) difficult. Progress has been made using various approximations such as assuming long waves (Killworth [1983] for a reduced gravity shallow water equation model), nearly uniform potential vorticity (Kubokawa [1985] again for a one-layer model) or long waves and deep lower layers (Killworth et al. [1984] for a two-layer isolated front). In the two-layer case these analytic solution techniques fail when the bottom topography varies in the cross-front direction.

Since little progress can be made analytically on the system (A4), a numerical solution technique is desirable. The coupled layer equations are cast in the form of an algebraic eigenvalue problem by splitting the cross-front domain into finite intervals and approximating the derivatives in (A4) by finite differences. The eigenvalues ($\lambda$) and eigenfunctions (cross-front structures) are then found by standard numerical techniques.

Boundary conditions for (A4) are as follows. One general requirement is that the solutions be frontally trapped. Far from the surface front on the two-layer side of the front ($x \rightarrow -\infty$) $\zeta_1$ and $\zeta_T$ vanish. On the one-layer side of the front, one of two boundary conditions is required. For an isolated front far from any coastal barrier the solution must again decay away from the front. However, the emphasis of this study is on the coastal upwelling front, which requires a coastal barrier for its existence. In this case, the numerical domain extends to the coast where the boundary condition of no normal flow is applied. The final boundary condition required to specify the problem uniquely is applied to the upper layer equation at the front. It was noted previously that the governing equation (A4a) was singular at $x = x_f$, so one way to express the boundary condition is that $\zeta_1$ re-
main finite at the front. This qualitative condition is made
quantitative [Barth, 1987] in order to implement it in a nu-
merical scheme by using the fact that the surface front must
be a material surface.

The actual solution of (A4) with the appropriate bound-
ary conditions proceeds by first specifying a basic state
flow. Choices are then made for the offshore distance to the
surface front ($x_f$), bottom topography, and/or lower layer
depth far offshore ($H_2$; see Figure 1). These choices com-
pletely specify the basic state and model geometry. Solu-
tions are then found for various alongfront wave numbers
($l$) to investigate the potential instability of the basic state
to perturbations of a given alongfront scale. The analyses
can then be repeated for different choices of the basic state
flow or model geometry.

When linearized bottom friction is included, the lower
layer geostrophic momentum equations for the perturba-
tions become

$$
\zeta_T + u_2 = \zeta_T - \frac{\lambda}{h_2} \zeta_T
$$

$$
\zeta_T + u_2 = -\zeta_T - \frac{\lambda}{h_2} \zeta_T
$$

where $\lambda$, the nondimensional friction coefficient, is related
to the dimensional version (which has units of velocity) by

$$
\lambda = f H_1
$$

Note that the friction velocities have been made geostrophic,
consistent with the geostrophic momentum approximation.
Substituting normal mode form for $\zeta_T$ (from (A1)), the
momentum equations become

$$
(\sigma + \frac{i \lambda}{h_2}) \zeta_T + u_2 = \zeta_T
$$

$$
-\left( \sigma - \frac{\lambda}{h_2} \right) \zeta_T + u_2 = -i \zeta_T
$$

Again, solving for the lower layer velocities and substituting
into the lower layer continuity equation (A3c) results in a
complex ordinary differential equation for $\zeta_T$

$$
\zeta_T + \frac{h_2}{h_2} \zeta_T - \left( \frac{1}{h_2} + i l \right) \zeta_T + \frac{1}{h_2} \zeta_1
$$

$$
= \frac{1}{\sigma} \left[ \frac{-i \lambda}{h_2} \zeta_T + \left( \frac{l h_2}{h_2} + \frac{i l \lambda}{h_2} \right) \zeta_T \right]
$$

The finite difference forms of this equation and the upper
layer equation (A4a) are again combined to yield a com-
plex algebraic eigenvalue problem. Boundary conditions at
the wall and at the offshore edge of the two-layer region are
modified by the presence of bottom friction. The appro-
priate boundary conditions [Barth, 1987] contribute terms
which are nonlinear in the eigenvalue. The problem is made
linear in the eigenvalue [following Webster, 1987] and then
solved using standard numerical techniques.

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